

Queen Mary and Westfield College

UNIVERSITY OF LONDON

MAS 204 Calculus III

1 June 1999 10.00-12.00

Time Allowed: 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination.

You are reminded of the following which you may use without proof

In orthogonal curvilinear coordinates (u_1, u_2, u_3) , with corresponding unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3 , the divergence of a vector field $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_3 h_1 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right],$$

and the curl of the same vector field is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$.

In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, z)$, the arc-length parameters are $h_1 = 1$, $h_2 = r$, $h_3 = 1$.

SECTION A: Each question carries 8 marks. You should attempt ALL questions.

A1. If $f = \sin(\pi xy) + yz^2$ and $\mathbf{F} = x^2\mathbf{i} - 3xy\mathbf{j} + 2xz\mathbf{k}$, evaluate the following:

(a) ∇f , (b) $\nabla \cdot \mathbf{F}$, (c) $\nabla \times \mathbf{F}$, (d) $\nabla^2 f$.

A2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3z\mathbf{i} + 5x^2\mathbf{j} - 2y\mathbf{k}$ and C is the path going from (0, 0, 0) to (1,1,1) described in parametric form by $C: \mathbf{r} = t^2\mathbf{i} + t^3\mathbf{j} + t\mathbf{k}$ with $0 \leq t \leq 1$.

A3. Using index notation, show that $\partial x_j / \partial x_i = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Hence or otherwise prove that

$$\nabla(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c}$$

where $\mathbf{r}$ is the position vector and $\mathbf{c}$ is a constant vector.

A4. Use the Divergence Theorem, or otherwise, to evaluate $\int \mathbf{F} \cdot \mathbf{n} dS$ over the surface of a sphere of radius a and centre (1, 1, 1), where $\mathbf{F} = 6x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}$ and $\mathbf{n}$ is a unit vector normal to the surface directed out of the sphere.

A5. Write down $\nabla^2 V$ in spherical polar coordinates (r, θ, ϕ) . Show that $V = r^{-2} \cos \theta$ is a solution of Laplace's equation, $\nabla^2 V = 0$.

A6. A fluid velocity is described by a vector field $\mathbf{u}$. Under what circumstances does $\mathbf{u}$ define (i) an irrotational flow and (ii) an incompressible flow? Derive an equation for the streamlines of the flow described by $\mathbf{u} = (kx, -ky, \mathbf{0})$ in a cartesian system, where k is a positive constant. Sketch the resulting streamlines.

A7. Maxwell's equations are given by

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho / \epsilon_0 & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\partial \mathbf{B} / \partial t & \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + (1/c^2) \partial \mathbf{E} / \partial t \end{aligned}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is the charge density, $\mathbf{j}$ is the current density, and $c^2 = 1/\epsilon_0 \mu_0$, ϵ_0 and μ_0 are all constants. Use Maxwell's equations to show that $\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0$.

SECTION B: Each question carries 22 marks. You may attempt all questions but only marks for the best TWO questions will be counted.

B1. State Stokes's theorem, defining your terms clearly. Using the field $\mathbf{F} = 2xy\mathbf{i} - 3yz\mathbf{j}$, show that Stokes's theorem is satisfied for the closed path in the $z = 0$ plane consisting of the four straight line segments from the origin to $(a, 0, 0)$ to $(a, b, 0)$ to $(0, b, 0)$, and back to the origin, where a and b are constants.

B2. Write down Laplace's equation, $\nabla^2 V = 0$, in cylindrical polar coordinates, (r, θ, z) , for a scalar field, V . Consider the special case where V is a single-valued function of r and θ only and let $V(r, \theta) = R(r)S(\theta)$ be a separable solution of Laplace's equation. Show that the functions R and S must satisfy

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = -\frac{1}{S} \frac{d^2 S}{d\theta^2} = m^2$$

where m is a non-negative integer. Hence deduce the general solution of Laplace's equation in cylindrical polar coordinates.

B3. Let C be a simple closed path in the x - y plane, traversed counter-clockwise, and let f and g be suitably differentiable functions. Show that

$$\int_C (f dx + g dy) = \int \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy$$

where the surface integral is taken over the region enclosed by the curve C . Use this result to evaluate $\int (xy^2 dy - y^3 dx)$ around the closed path formed by the straight line segments from the origin to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ and back to the origin.

B4. Legendre's equation,

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \lambda y = 0$$

where $x = \cos \theta$, arises in the study of separable solutions of Laplace's equation in spherical polar coordinates. Use the method of Frobenius to show that Legendre's equation has a convergent, regular solution if $\lambda = 1(1 + 1)$, where 1 is a positive integer.